Pedagogical content knowledge of preservice mathematics teachers: An analysis of classroom observations

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Observations of 104 lessons taught by final-year secondary mathematics student teachers were analysed to identify weaknesses in their pedagogical content knowledge. "Pedagogical mathematics weaknesses" were found in the areas of meaning and purpose, accuracy and appropriateness, quality of explanation and quality of language. Six "pedagogical mathematics themes" were identified as in need of particular attention. It is argued that teacher education programs need to include more explicit discussion of the purpose, basic concepts, and pupil learning of such central mathematical themes.

There has recently been an upsurge of interest in mathematics teachers' subject-matter knowledge. It seems intuitively obvious that the quality of teachers' understanding should affect their teaching and hence the mathematics learning of their pupils, but earlier research in the "effective teaching" tradition had failed to find any relation between teacher knowledge and pupil performance. Shulman's (1986) differentiation between content knowledge, pedagogical content knowledge ("subject matter knowledge *for teaching*") and curricular knowledge (knowledge of "the full range of programs designed for the teaching of particular subjects and topics at a given level") seems to have revived interest in what had appeared to be a dead horse. Recent attention has focussed on pedagogical content knowledge, which includes "what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (Shulman, 1986, p. 9).

Most of the research on pedagogical content knowledge has focussed on preservice elementary teachers. In this paper, I shall deal only with preservice secondary mathematics teachers. In order to make a clear differentiation, I shall refer to student teachers as "students" and school students as "pupils."

Ball (1990a, 1990b) has studied understanding of division of fractions among students just enrolling in several teacher education programs. In one case, 35 students were asked to select which of three story problems illustrated what $4\ 1/4 \div 1/2$ means. An astonishing 60% did not select the appropriate problem or indicated that they did not know which one was appropriate. In another case, two samples of students were asked in individual interviews to generate representations of $1\ 3/4 \div 1/2$. In one sample 6 out 10 and in another 4 out of 9 either generated an inappropriate representation or could not generate any representation. In further questions of the second sample, 4 of the 9 students could not explain (as if to their pupils) what $7 \div 0$ is and *none* could suggest any way to help pupils solve $x \div 0.2 = 5$ apart from describing a mechanical solution procedure. Such results make it clear that students entering teacher education programs may have very poor pedagogical content knowledge even if (as was the case with several of Ball's subjects) they have achieved excellent mathematics results at school.

Students' conceptions of function have been studied by Even (1990, 1993), Norman (1993) and White & Mitchelmore (in press). Even has developed a framework of seven aspects of pedagogical content knowledge which can be applied to any mathematical concept. In a study of 162 student teachers in the last stage of their formal preservice preparation, Even (1990) found—despite majors in mathematics—"a limited concept of function, similar to one from the 18th century" (p. 528) and a complete inability to explain the importance of univalence [every element of the domain of a function has a unique image in the range] or the reason why functions came to be defined as univalent. Students were also generally weak on relating different representations of functions (e.g. symbolic and graphical), using different approaches to investigating functions (e.g pointwise versus global), comprehension of inverse functions, and knowledge of the basic functions studied in school.

Norman (1993) presents no new empirical evidence, but draws on results on teacher knowledge in general and on school students' understanding of algebra to predict aspects of functions and graphs which might not be well understood by teachers and to formulate a long list of related research questions.

White & Mitchelmore (in press) found that first year preservice teachers were severely hampered in their learning of calculus by an immature concept of variable. The majority of the students seemed to treat variables as symbols to be manipulated rather than quantities to be related, which again seems to point to a poor concept of function.

The only other study of secondary student teachers' pedagogical content knowledge which I have been able to locate investigated their understanding of proportion (Fisher, 1988). Ball (1990a) mentions further studies of perimeter and area, proportion, variables and solving equations, theory and proof, slope and graphing, but the results do not appear to have been published. There is clearly a need for more research in this area.

The Present Study

I have been supervising final year students at Macquarie University since 1991 and have accumulated notes of a large number of lesson observations. In my notes, I have frequently recorded for subsequent discussion with the student teacher specific mathematical strengths and weaknesses as well as aspects of general classroom management. It was felt that an analysis of these notes could make a useful contribution to our understanding of preservice mathematics teachers' pedagogical content knowledge. In contrast to previous studies, which ask students how they would hypothetically explain mathematical topics, these notes indicate what students actually do in the classroom.

It is as well to admit at the outset that such observations are subject to strong observer bias. What I consider to be a weakness might not even have been noted by observers with a different philosophical stance. However, I believe that my stance—being based on a commitment to constructivism as an epistemology and to cognitively guided instruction as a model for teacher education—is shared by most mathematics educators today. In fact, the bias is probably no greater than that exercised by a researcher in designing a more "objective" observation instrument. A greater limitation lies in the fact that observations were not systematic, as attention was often diverted to other questions which are equally important in supervising student teachers. Nevertheless, it is anticipated that an analysis of the observations could provide some pointers to the major areas of weakness and hence be of value in designing teacher education programs.

Method

The first step was to review my records of the 104 lessons observed between 1991 and 1994. These lessons had been taught by a total of 39 male and 43 female students and covered all content categories (arithmetic, algebra, functions and graphs, approximation and measurement, statistics and probability, and geometry and trigonometry) and all ages (years 7-12). I first noted all remarks (both positive and negative) which specifically related to students' understanding of mathematics, to the way they presented mathematics to pupils, or to their understanding of how pupils learn mathematics. The result was a list of 208 notes, 190 of which indicated what I thought were weaknesses. For lack of a better term, I shall refer to these 190 occurrences as *pedagogical mathematics weaknesses*.

The second step was to classify the 190 pedagogical mathematics weaknesses into general aspects of pedagogical content knowledge. After some trial and error, four categories were decided on: meaning and purpose, accuracy and appropriateness, quality of explanation, and quality of language. Each category had several subcategories.

A further categorisation, intended to identify the major mathematical themes in which pedagogical mathematics weaknesses occurred, was made after the initial classification and will be described below.

Results

Table 1 presents the results of the initial classification. It will be seen that the pedagogical mathematics weaknesses were spread over all six content categories, but were slightly more frequent in algebra and approximation and measurement, and rather less frequent in statistics and probability.

Table 1 Distribution of pedagogical mathematics weaknesses by type and content area

Type of weakness	Arith.	Alg	Funct& Graph	Approx . & Meas.	Stats. & Prob.	Geom. & Trig.	Total
Meaning & purpose	10	6	12	7	5	4	44
Accuracy & appropriateness	5	5	10	11	9	4	44
Quality of explanation	15	17	24	19	10	3	88
Quality of language	2	4	7	0	1	0	14
Total	32	32	53	37	25	11	190
No. of lessons	17	13	30	16	15	13	104
Weaknesses per lesson	1.9	2.5	1.8	2.3	1.7	0.8	1.8

Although it is not obvious from Table 1, the most frequent type of pedagogical mathematics weakness was "meaning and purpose." In only 39 lessons was an explicit note made that the student teacher had not given any reason for being interested in the subject of the lesson, but a review of my notes suggested that this was also the case in most of the other lessons. The notes only indicate seven cases where some purpose was made explicit.

The next largest category was "quality of explanations". The most frequently occurring subcategories were as follows:

- Not giving an explanation in class or not being able to give an explanation after the class (31 occurrences). For example, several instances were observed of describing a geometrical construction procedure without any attempt to justify it; stating a geometrical result without relating it to any other result; and stating arithmetical rules without explanation or being unable to explain them.
- Giving a misleading or incomplete explanation (15 occurrences). Typical examples were use of the fruit salad analogy in algebra, explaining a x2

enlargement as an extension by an equal distance, and defining a function without any mention of relationships.

- Omitting crucial steps, on the assumption that pupils would see the connection without help (12 occurrences). Typical examples were transforming $C=\pi d$ to $C=2\pi r$ without explanation and equating $-x$ to $(-1)x$.

Some other weak explanations failed to make a link which might have assisted students to understand a new concept or result, relied on a verbal formalisation of a result, or failed to make an easy check.

The third most frequent pedagogical mathematics weakness was the accuracy and appropriateness of students' mathematics (44 occurrences). There were 35 cases of mathematical error and a number of cases where students used mathematically poor (but not wrong) methods. Some examples:

- Using the distance formula and Pythagoras' theorem in coordinate geometry as if they were independent results.
- Stating that to find a limit as $h \rightarrow 0$ "you put $h=0$ in the resulting expression".
- Stating that all approximations should be made to 1 d.p.
- Not rounding off the results of measurement calculations.

Most of the pedagogical mathematics weaknesses in the category "quality of language" fell under the heading of sloppy language. Half of the examples occurred in geometry lessons where some typical instances were to accept a pupil's "These letters add up to 180°" and to use "side" to refer to both edges and faces of 3D figures.

The observed pedagogical mathematics weaknesses seemed to be randomly strewn over all content categories. However, a closer examination showed that the great majority of them fell into six broad themes. I shall refer to these as *pedagogical mathematics themes* (see Table 2).

Table 2 Distribution of pedagogical mathematics weaknesses, by pedagogical mathematics theme

Pedagogical mathematics theme	N
1. Conceptual basis for fractions, decimals, percentages, ratio and the corresponding manipulation rules	26
2. Numbers in measurements, accuracy of measurements and results of measurement calculations	15
3. Algebra as a conventional language for summarising numerical properties and relations	31
4. Algebra and graphs as problem-solving tools	33
5. The empirical basis for geometrical concepts and relations, practical applications of geometry	40
6. The logical basis for deductive geometry	18
Other	27

The first two pedagogical mathematics themes are closely related and essentially amount to an awareness of the basic structure, purpose and usefulness of arithmetic. The third and fourth themes are also closely related and represent the conceptual basis for algebra and the reason for its usefulness respectively. The pedagogical mathematics weaknesses in these two themes were found in several content categories—not only in algebra and in functions and graphs but also in statistics and probability and in coordinate geometry. The final two themes represent two distinct aspects of school geometry: empirical-inductive and logical-deductive. Student teachers seemed to be rather weak in their knowledge both of how pupils form basic spatial concepts and of how mathematics builds them into a logical system—and they did not clearly differentiate these two aspects.

Discussion

One might argue that an average of 1.8 "pedagogical mathematics weaknesses" per lesson is not a huge number. However, when one takes into account the fact that the meaningless of most lessons observed was frequently not explicitly noted and that the observations do not represent any systematic attempt to record weaknesses, the data are suggestive of some deep underlying problems.

Most of the pedagogical mathematics weaknesses seem to derive from a belief on the part of students that mathematics is a set of definitions and results to be learnt rather than a set of concepts and relations which help to explain and simplify. This seems clear both in students' almost universal failure to justify the topics of their lessons and in their frequent failure to explain results or to link different results together. They did not seem to credit pupils with any need to know why they are studying mathematics or to make sense of what they are learning. In many cases, it would seem that students were themselves not aware of the purpose of the mathematics or of the links between different topics.

It could be argued that the pedagogical mathematics weaknesses revealed by the present study are not typical of preservice secondary mathematics teachers in Australia. However, Macquarie's teacher education program has been judged as exemplary (Department of Employment, Education and Training, 1989) so its students are probably not less well qualified or educated than teacher education students at other universities. It might also be argued that a BEd model, where mathematics content is taught by the education department rather than the mathematics department, should lead to greater pedagogical content knowledge. Certainly the potential is there, but there is no evidence that that potential is realised. In fact, there is very little hard evidence at all about Australian secondary mathematics teachers' pedagogical content knowledge.

If the general pattern of results from the present study is confirmed at other sites, then the implication is that students are not learning sufficient pedagogical content knowledge from current teacher education programs. They certainly do not learn it from their formal mathematics studies, as Ball (1990a) has emphasised, but then that is not among the aims of the average mathematics course. More disappointingly, they do not appear to acquire adequate pedagogical content knowledge from their education course—despite the recent emphasis on constructivism and problem-solving and on changing students' beliefs about the nature of mathematics. Such courses may indeed change students' beliefs; for example, one unit in the Macquarie program has been shown to significantly change student teachers' beliefs towards a more open view of mathematics (Mitchelmore, 1992). However, this change in belief does not seem to be having the desired effect on student teachers' actions in the classroom. Either students are only making a show for the sake of pleasing their lecturers and getting good grades, or they are not being given sufficient guidance in how to translate their new ways of looking at mathematics into appropriate classroom practices.

It seems likely that a teacher education program would be more effective in bringing about significant pedagogical content knowledge if it were directed at more specific themes rather than at some general change in belief about mathematics. The six pedagogical mathematics themes outlined in Table 2 provide a possible framework. A serious study of a carefully selected small sample of school mathematics topics within each theme, exploring their purpose and usefulness, their conceptual basis and their links to other concepts, could provide student teachers not only with examples which they could use immediately but also with models on which to base their teaching of other topics.

References

Ball, D. L. (1990a). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90,* 449-466.

Ball, D. L. (1990b). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21,* 132-134.

Department of Employment, Education and Training (1989). *Discipline review of teacher education in mathematics and science.* Canberra: Australian Government Publishing Service.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics,* 21, 521-544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24,* 94-116.

Fisher, L. C. (1988). Strategies used by secondary school mathematics teachers to solve proportion problems. *Journal for Research in Mathematics Education,* 19, 157-168.

Mitchelmore, M. C. (1992). *Increasing preservice primary teachers' mathematical confidence: an Australian experience.* Paper presented to Working Group on Preservice and Inservice Teacher Education, 7th International Congress on Mathematical Education, Quebec.

Norman, F. A. (1993). Integrating research on teachers' knowledge of functions and their graphs. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 159-187). Hillsdale, NJ: Lawrence Erlbaum.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

White, P., & Mitchelmore, M. C. (in press). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education.*